Preliminary Note on the Operational Invariants of a Binary Quantic.

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My object is to present a theory of operational invariants for the binary quantic of order n. Many of the results that are here given have been explicitly or implicitly given in the works of Cayley, Clebsch, Gordan, and other writers, but not from the present point of view or in the convenient notation that will be adopted. Some quite new results are reached, which will be found to throw much light upon the work of others.

1. In the algebraic theory we take the quantic to be

$$a_x^n = (a_1x_1+a_2x_2)^n,$$

and $b, c, d, \ldots$ as alternative symbolic letters to a. We then have symbolic factors of two types, viz., the suffix type, a_x, and the bracketed or determinant type (ab). In fact, for the substitution

$$x_1 = \lambda_1 X_1+\mu_1 X_2,$$
$$x_2 = \lambda_2 X_1+\mu_2 X_2,$$

the determinant of which is

$$\lambda_1\mu_2-\lambda_2\mu_1 = (\lambda\mu) = \Delta,$$

we have
$$a_x^n = A_x^n = (A_1X_1+A_2X_2)^n;$$
where
$$A_1 = a_1\lambda_1+a_2\lambda_2 = a_\lambda,$$
$$A_2 = a_1\mu_1+a_2\mu_2 = a_\mu,$$
$$(ab) = \Delta^{-1}(AB).$$

The real form of a_x^n is

$$a_0x_1^n+\binom{n}{1}a_1x_1^{n-1}x_2+\binom{n}{2}a_2x_1^{n-2}x_2^2+\ldots+a_nx_2^n.$$

I write symbolically
$$\partial_{a_s} = \binom{n}{s}\alpha_1^{n-s}\alpha_2^s,$$
$$\frac{\partial^{p+q}}{\partial^p_{x_1}\partial^q_{x_2}} = \xi_1^p\xi_2^q;$$

whence it is readily seen that α_1, α_2
are cogredient to x_1, x_2;
and that ξ_1, ξ_2
are cogredient to a_1, a_2;

and that thus we obtain the new invariant symbolic factors

$$(x\alpha),\ (\alpha\beta),\ (a\xi),\ (\xi\eta),$$

$$x_\xi,\quad \alpha_\xi,\quad a_\alpha,\quad a_\xi.$$

To make this clear, I write the symbolic letters in two rows—

$$\begin{array}{ll} x,\ y,\ z,\ \ldots\ldots & \alpha,\ \beta,\ \gamma,\ \ldots\ldots, \\ a,\ b,\ c,\ \ldots\ldots & \xi,\ \eta,\ \zeta,\ \ldots\ldots, \end{array}$$

where $x, y, z, \ldots, \alpha, \beta, \gamma, \ldots, a, b, c, \ldots, \xi, \eta, \zeta, \ldots$ are alternative letters respectively, and remark that all letters in the same row are cogredient, and that letters which are in different rows are contragredient. Determinant invariant symbolic factors are obtained from any two letters in the same row, and suffix invariant symbolic factors from any two letters in different rows.

An invariant factor corresponds to *every* pair of letters, so that if there be p different letters, there will be

$$\binom{p}{2}$$

invariant symbolic factors.

As regards any two letters in the first row, the substitution gives

$$(xy) = \Delta(\mathrm{XY}),$$

the power of the modulus for transformation being unity.

For any two letters in the second row

$$(a\xi) = \Delta^{-1}(\mathrm{A}\Xi),$$

the power of the modulus being negative unity. For any two letters in different rows the invariance is absolute, *e.g.*, $x_\xi = \mathrm{X}_\Xi$.

In a symbolic product absolute invariance is secured when the number of determinant factors drawn from the first row is equal to the number drawn from the second. The power of the modulus is always equal to the excess of the number of such factors drawn from the first row over the number drawn from the second.

We are now able to form a calculus of operational invariants. All the operational processes at present in existence, including those which produce evectants, provectants, and emanants, the Aronhold process, the Ω process, and transvection, are included in the present enlarged symbolism. An invariant operator may or may not involve the variables x_1, x_2, and it may be separable into two portions, one of which involves the variables, while the other does not. The different classes will present themselves for examination as the work proceeds. Those operators which do not involve ξ_1 and ξ_2 may be homogeneous in the variables, and are then analogous to algebraical covariants. The leading coefficient will then be found to have semi-invariant properties, and, of course, the same also may be said of the end coefficients.

2. First consider the covariant $(x\alpha)^n$ of the quantic a_x^n. In unsymbolic form it is

$$(\partial_{a_n}, -\partial_{a_{n-1}}, +\partial_{a_{n-2}}, \ldots \between x_1, x_2)^n,$$

or

$$x_1^n\partial_{a_n} - x_1^{n-1}x_2\partial_{a_{n-1}} + \ldots (-)^n x_2^n \partial_{a_0},$$

a covariant of order n and of degree -1, weight $-n$ in the coefficients. Operating upon any algebraical covariant, it produces a covariant, and it is easy to see that, in many cases, it causes the covariant to vanish. For, if the operand be of

$$(\text{deg., weight, order}) = (i, w, \epsilon),$$

it is transformed into one of

$$(\text{deg., weight, order}) = (i-1, w-n, \epsilon+n),$$

and vanishes for the cases

$$i = 1, \; w < n, \; w = n+1.$$

Annihilation takes place when the number of determinant factors in the operand is $n+1$ or $< n$.

In particular, a_x^n is annihilated and $(n > 2)$, so is the Hessian; and so on.

In general, the symbolic factor $(x\alpha)$ is concerned with the process known as evection.

3. Next consider the covariant operator

$$a_x^p (x\alpha)^p \alpha_\alpha^{n-p},$$

which is lineo-linear in the coefficients, where p may be any one of the integers $0, 1, 2, \ldots n-1$.

It has $\qquad (\text{deg., weight, order}) = (0, -p, 2p),$

and therefore converts any covariant of characters (i, w, ϵ) into one which has characters $(i, w-p, \epsilon+2p)$.

It is obvious that it annihilates any covariant operand for which $w < p$ or $w = p+1$. It is merely necessary to observe that the coefficient of $x_1^{\epsilon+2p}$ vanishes, and this implies the vanishing of the form. Observe also that if $i = 2$, the form vanishes unless $w-p$ is even.

The case $p = 0$ gives the single invariant operator which is lineo-linear in the coefficients, viz.,

$$\alpha_\alpha^n = \mathrm{I}_a = a_0\partial_{a_0} + a_1\partial_{a_1} + \ldots + a_n\partial_{a_n}.$$

In order to exhibit to the best advantage the unsymbolic forms of these operators, I write, following as far as possible the notation adopted by other writers,

$$\Omega_1 = a_0\partial_{a_1}+2a_1\partial_{a_2}+\ldots+na_{n-1}\partial_{a_n},$$

$$\Omega_p = a_0\partial_{a_p}+\binom{p+1}{p}a_1\partial_{a_{p+1}}+\binom{p+2}{p}a_2\partial_{a_{p+2}}+\ldots+\binom{n}{p}a_{n-p}\partial_{a_n},$$

$$\mathrm{O}_1 = a_n\partial_{a_{n-1}}+2a_{n-1}\partial_{a_{n-2}}+\ldots+na_1\partial_{a_0},$$

$$\mathrm{O}_p = a_n\partial_{a_{n-p}}+\binom{p+1}{p}a_{n-1}\partial_{a_{n-p-1}}+\binom{p+2}{p}a_{n-2}\partial_{a_{n-p-2}}+\ldots+\binom{n}{p}a_p\partial_{a_0};$$

where, to explain, if $\dagger$ be a symbol to denote operation upon the symbols of quantity only, the differential inverses being regarded as constants,

$$\frac{1}{2!}\Omega_2 = \Omega_1\dagger\Omega_1,$$

$$\frac{1}{3!}\Omega_3 = \Omega_1\dagger\Omega_1\dagger\Omega_1,$$

. . . .

$$\frac{1}{2!}\mathrm{O}_2 = \mathrm{O}_1\dagger\mathrm{O}_1,$$

$$\frac{1}{3!}\mathrm{O}_3 = \mathrm{O}_1\dagger\mathrm{O}_1\dagger\mathrm{O}_1,$$

. . . . . .

I further write

$$\mathrm{W} = \mathrm{W}_1 = a_1\partial_{a_1}+2a_2\partial_{a_2}+\ldots+na_n\partial_{a_n},$$

$$\mathrm{W}_p = a_1\partial_{a_1}+2^pa_2\partial_{a_2}+\ldots+n^pa_n\partial_{a_n};$$

where, to explain the formation,

$$\mathrm{W}_2 = \mathrm{W}_1\dagger\mathrm{W}_1,$$

$$\mathrm{W}_3 = \mathrm{W}_1\dagger\mathrm{W}_1\dagger\mathrm{W}_1,$$

. . . .

Observe that the notation gives

$$\mathrm{I} = \Omega_0 = \mathrm{O}_0 = \mathrm{W}_0.$$

I further find it convenient to write symbolically

$$\mathrm{W}_p = \mathrm{W}^p,$$

so that

$$\tfrac{1}{2}(\mathrm{W}_2-\mathrm{W}_1) = \binom{\mathrm{W}}{2},$$

$$\tfrac{1}{6}(\mathrm{W}_3-3\mathrm{W}_2+2\mathrm{W}_1) = \binom{\mathrm{W}}{3}.$$

. . . .

4. When $p = 1$, I find, for $a_x(x\alpha)a_\alpha^{n-1}$, the expression

$$x_1^2\Omega_1-x_1x_2(n\mathrm{I}-2\mathrm{W})-x_2^2\mathrm{O}_1,$$

which merits a careful study, as it is the simplest invariantive form which involves the well-known seminvariant operators Ω_1, O_1.

From the known properties of Ω_1, $n\mathrm{I}-2\mathrm{W}$, and O_1 it is clear that it annihilates every invariant of a_x^n.

Also Ω_1 annihilates the leading coefficient of every covariant. Hence, when a covariant ϕ_x^ϵ is the operand, the term involving $x_1^{\epsilon+2}$ is absent, and, as a result, the operator annihilates also every covariant of a_x^n.

Next, in the operator, put $x_1 = x_2 = 1$. We see that the sum of the coefficients of any covariant satisfies the equation

$$\Omega_1-(n\mathrm{I}-2\mathrm{W})-\mathrm{O}_1 = 0.$$

The isobaric solutions of this equation are invariants. The symbolic form of this operator is

$$(a_1+a_2)(\alpha_2-a_1)\,a_\alpha^{n-1};$$

it annihilates

$$(\phi_1+\phi_2)^\epsilon,$$

if ϕ_x^ϵ be any covariant.

Any covariant $$\phi_x^\epsilon = (\phi_0,\ \phi_1,\ \phi_2),\ \ldots \between x_1,\ x_2)^\epsilon$$

has the property that any three consecutive coefficients,

$$\phi_s,\ \phi_{s+1},\ \phi_{s+2},$$

satisfy the relation

$$\Omega_1\phi_{s+2}-(n\mathrm{I}-2\mathrm{W})\,\phi_{s+1}-\mathrm{O}_1\phi_s = 0,$$

wherein ϕ, with a negative suffix or with a suffix greater than ϵ, is to be put equal to zero.

It will be noted that the operators Ω_1, O_1 from their positions in the covariant are themselves seminvariants.

Next take the Hessian of the covariant operator, viz.,

$$4\Omega_1\mathrm{O}_1+(n\mathrm{I}-2\mathrm{W})^2,$$

where the multiplications are purely algebraical and not at all operational. We have thus an invariant operator of the second order, which will come under view later on in connection with Cayley's operators

$$x_2\partial_{x_1}-\Omega_1,\ \ x_1\partial_{x_2}-\mathrm{O}_1.$$

5. I pass to the case $p = 2$, which has the unsymbolic form

$$\begin{gathered}
x_1^4\Omega_2 \\
-x_1^3x_2\left\{\binom{n-1}{1}\mathrm{I}-2\mathrm{W}\right\}\Omega_1 \\
+x_1^2x_2^2\left\{\binom{n}{2}\mathrm{I}-3\binom{n-1}{1}\mathrm{W}+6\binom{\mathrm{W}}{2}\right\} \\
-x_1x_2^3\mathrm{O}_1\left\{\binom{n-1}{1}\mathrm{I}-2\mathrm{W}\right\} \\
+x_2^4\mathrm{O}_2,
\end{gathered}$$

wherein the multiplications of operators are such that the left-hand operator is performed merely upon the literal coefficients of the right-hand operator, and not at all upon the differential inverses.

Ex.—
$$\begin{aligned} W_1\Omega_1 &= (a_1\partial_{a_1}+2a_2\partial_{a_2}+\ldots)(a_0\partial_{a_1}+2a_1\partial_{a_2}+\ldots) \\ &= 1\,.\,2\,a_1\partial_{a_2}+2\,.\,3\,a_2\partial_{a_3}+\ldots, \\ O_1W_1 &= (a_n\partial_{a_{n-1}}+2a_{n-1}\partial_{a_{n-2}}+\ldots)(a_1\partial_{a_1}+2a_2\partial_{a_2}+\ldots) \\ &= 1\,.\,n-1\,.\,a_n\partial_{a_{n-1}}+2\,.\,n-2\,.\,a_{n-1}\partial_{a_{n-2}}+\ldots. \end{aligned}$$

I note first that the operator always produces a covariant.

Ω_2 and O_2 are seminvariants. Ω_2, for example, either annihilates a leading seminvariant or converts it into another one. In the former case the operator covariant causes the operand covariant to vanish; for

$$\Omega_2 = a_0\partial_{a_2}+3a_1\partial_{a_3}+ba_2\partial_{a_4}+\ldots,$$

and the alternant of Ω_1 and Ω_2 does in fact vanish.

What seminvariants are caused to vanish by this operator Ω_2? The reply to this question is supplied by the theory of symmetric functions. For make the transformation

$$a_s = s\,!\,b_s,$$

so that the seminvariants are non-unitary symmetric functions of the roots of the equation

$$b_0x^m - b_1x^{m-1}+b_2x^{m-2}-\ldots = 0;$$

Ω_2 is equivalent to $\quad b_0\partial_{b_2}+b_1\partial_{b_3}+b_2\partial_{b_4}+\ldots,$

and this again to $\quad D_1^2-2D_2,$

where $$D_\kappa = \frac{1}{\kappa\,!}(b_0\partial_{b_1}+b_1\partial_{b_2}+b_2\partial_{b_3}+\ldots)^\kappa.$$

Take a non-unitary symmetric function—

$$\Sigma\alpha^{k_1}\beta^{k_2}\gamma^{k_3}\ldots = (k_1k_2k_3\ldots);$$

then $$D_{k_1}(k_1k_2k_3\ldots) = (k_2k_3\ldots),$$

and D_k annihilates every symmetric function whose partition contains no part k.

Hence Ω_2 here is equivalent to D_2 and all symmetric functions which, when expressed in partitions, contain no part 2 are solutions of the equation

$$\Omega_2 = 0.$$

E.g., the covariant $\quad (a_0^2a_3-3a_0a_1a_2+2a_1^3)x_1^\epsilon+\ldots$

is annihilated by the quantic covariant operator, because $a_0^2a_3-3a_0a_1a_2+2a_1^3$, when transformed, is the symmetric function $\Sigma\alpha^3$ or (3), the number 2 not appearing in the partition. The covariant of deg. 3 and weight 5 in the coefficient is not annihilated because the corresponding partition is (32); we learn, moreover, that the result of the operation is the covariant of deg. weight (3, 3) which corresponds to the partition (3).

Generally the covariant $(ab)^{2s}a_x^s b_x^s$

is converted into $(ab)^{2s-2}a_x^{s+2}b_x^{s+2}$.

The operator, for the quantic, is, on development,

$$\begin{aligned}
&x_1^4(a_0\partial_{a_2}+3a_1\partial_{a_3}+6a_2\partial_{a_4})\\
&-x_1^3x_2(3a_0\partial_{a_1}+2a_1\partial_{a_2}-3a_2\partial_{a_3}-12a_3\partial_{a_4})\\
&+3x_1^2x_2^2(2a_0\partial_{a_0}-a_1\partial_{a_1}-2a_2\partial_{a_2}-a_3\partial_{a_3}+2a_4\partial_{a_4})\\
&-x_1x_2^3(-3a_4\partial_{a_3}-2a_3\partial_{a_2}+3a_2\partial_{a_1}+12a_1\partial_{a_0})\\
&+x_2^4(a_4\partial_{a_2}+3a_3\partial_{a_1}+6a_2\partial_{a_0}),
\end{aligned}$$

of which the symmetry does not escape attention.

6. When $p = 3$ we obtain an annihilator, which has the form

$$\begin{gathered}
x_1^6\Omega_3\\
-x_1^5x_2\left\{\binom{n-2}{1}\mathrm{I}-2\mathrm{W}\right\}\Omega_2\\
+x_1^4x_2^2\left\{\binom{n-1}{2}\mathrm{I}-\tfrac{5}{2}\binom{n-2}{1}\mathrm{W}+5\binom{\mathrm{W}}{2}\right\}\Omega_1\\
-x_1^3x_2^3\left\{\binom{n}{3}\mathrm{I}-4\binom{n-1}{2}\mathrm{W}+10\binom{n-2}{1}\binom{\mathrm{W}}{2}-20\binom{\mathrm{W}}{3}\right\}\\
-x_1^2x_2^4\mathrm{O}_1\left\{\binom{n-1}{2}\mathrm{I}-\tfrac{5}{2}\binom{n-2}{1}\mathrm{W}+5\binom{\mathrm{W}}{2}\right\}\\
+x_1x_2^5\mathrm{O}_2\left\{\binom{n-2}{1}\mathrm{I}-2\mathrm{W}\right\}\\
-x_2^6\mathrm{O}_3,
\end{gathered}$$

operative for all quantics of order not less than 4.

7. The unsymbolic form of the operator

$$a_x^p(x\alpha)^p a_\alpha^{n-p}$$

is

$$\begin{aligned}
&x_1^{2p}\Omega_p\\
&-x_1^{2p-1}x_2\left\{\binom{n-p+1}{1}\mathrm{I}-\frac{\binom{2p}{1}}{\binom{p}{1}}\mathrm{W}\right\}\Omega_{p-1}\\
&+x_1^{2p-2}x_2^2\left\{\binom{n-p+2}{2}\mathrm{I}-\frac{\binom{2p-1}{1}}{\binom{p-1}{1}}\binom{n-p+1}{1}\mathrm{W}+\frac{\binom{2p}{2}}{\binom{p}{2}}\binom{\mathrm{W}}{2}\right\}\Omega_{p-2}\\
&+\dots\dots
\end{aligned}$$

$$
\begin{aligned}
&+(-)^{s}x_1^{2p-s}x_2^{s}\Bigg\{\binom{n-p+s}{s}\mathrm{I}-\frac{\binom{2p-s+1}{1}}{\binom{p-s+1}{1}}\binom{n-p+s-1}{s-1}\mathrm{W}\\
&\qquad\qquad+\frac{\binom{2p-s+2}{2}}{\binom{p-s+2}{2}}\binom{n-p+s-2}{s-2}\binom{\mathrm{W}}{2}-\ldots+(-)^{s}\frac{\binom{2p}{s}}{\binom{p}{s}}\binom{\mathrm{W}}{s}\Bigg\}\Omega_{p-s}\\
&+\ldots\ldots\\
&+(-)^{p}x_1^{p}x_2^{p}\Bigg\{\binom{n}{p}\mathrm{I}-\frac{\binom{p+1}{1}}{\binom{1}{1}}\binom{n-1}{p-1}\mathrm{W}+\frac{\binom{p+2}{2}}{\binom{2}{2}}\binom{n-2}{p-2}\binom{\mathrm{W}}{2}-\ldots\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+(-)^{p}\binom{2p}{p}\binom{\mathrm{W}}{p}\Bigg\}\\
&+\ldots\ldots\\
&+(-)^{p+s}x_1^{s}x_2^{2p-s}\mathrm{O}_{p-s}\Bigg\{\binom{n-p+s}{s}\mathrm{I}-\frac{\binom{2p-s+1}{1}}{\binom{p-s+1}{1}}\binom{n-p+s-1}{s-1}\mathrm{W}\\
&\qquad\qquad+\frac{\binom{2p-s+2}{2}}{\binom{p-s+2}{2}}\binom{n-p+s-2}{s-2}\binom{\mathrm{W}}{2}-\ldots+(-)^{s}\frac{\binom{2p}{s}}{\binom{p}{s}}\binom{\mathrm{W}}{s}\Bigg\}\\
&+\ldots\ldots\\
&+(-)^{p}x_1^{2}x_2^{2p-2}\mathrm{O}_{p-2}\Bigg\{\binom{n-p+2}{2}\mathrm{I}-\frac{\binom{2p-1}{1}}{\binom{p-1}{1}}\binom{n-p+1}{1}\mathrm{W}+\frac{\binom{2p}{2}}{\binom{p}{2}}\binom{\mathrm{W}}{2}\Bigg\}\\
&+(-)^{p+1}x_1x_2^{2p-1}\mathrm{O}_{p-1}\Bigg\{\binom{n-p+1}{1}\mathrm{I}-\frac{\binom{2p}{1}}{\binom{p}{1}}\mathrm{W}\Bigg\}\\
&+(-)^{p}x_2^{2p}\mathrm{O}_{p}.
\end{aligned}
$$

It will be observed that, reading the rows down the page, the signs are alternately positive and negative as far as the middle row (involving $x_1^p x_2^p$) inclusive; after that row the signs are alternately negative and positive until the last row, which thus has the sign $(-)^p$.

It is operative for all quantics of order not less than $p+1$.

The Operators

$$x_2\partial_{x_1}-\Omega_1,$$
$$x_1\partial_{x_2}-\mathrm{O}_1.$$

8. Cayley, in his introductory memoir upon Quantics, defined a covariant of

a_x^n as a form which is annihilated by each of these operators. By the linear transformation are obtained the formulæ

$$\begin{aligned}\Delta(x_2\xi_1-\Omega_a) &= \mu_2^2(\mathrm{X}_2\Xi_1-\Omega_{\mathrm{A}})\\ &\quad+\lambda_2\mu_2\{\mathrm{X}_1\Xi_1-\mathrm{X}_2\Xi_2-(n\mathrm{I}_{\mathrm{A}}-2\mathrm{W}_{\mathrm{A}})\}\\ &\quad-\lambda_2^2(\mathrm{X}_1\Xi_2-\mathrm{O}_{\mathrm{A}}),\\ \Delta\{x_1\xi_1-x_2\xi_2-(n\mathrm{I}_a-2\mathrm{W}_a)\} &= 2\mu_1\mu_2(\mathrm{X}_2\Xi_1-\Omega_{\mathrm{A}})\\ &\quad+(\lambda_1\mu_2+\lambda_2\mu_1)\{\mathrm{X}_1\Xi_1-\mathrm{X}_2\Xi_2-(n\mathrm{I}_{\mathrm{A}}-2\mathrm{W}_{\mathrm{A}})\}\\ &\quad-2\lambda_1\lambda_2(\mathrm{X}_1\Xi_2-\mathrm{O}_{\mathrm{A}}),\\ \Delta(x_1\xi_2-\mathrm{O}_a) &= -\mu_1^2(\mathrm{X}_2\Xi_1-\Omega_{\mathrm{A}})\\ &\quad-\lambda_1\mu_1\{\mathrm{X}_1\Xi_1-\mathrm{X}_2\Xi_2-(n\mathrm{I}_{\mathrm{A}}-2\mathrm{W}_{\mathrm{A}})\}\\ &\quad+\lambda_1^2(\mathrm{X}_1\Xi_2-\mathrm{O}_{\mathrm{A}}),\end{aligned}$$

from which it is clear that a third annihilator of all covariants, viz.,

$$x_1\partial_{x_1}-x_2\partial_{x_2}-(n\mathrm{I}-2\mathrm{W}),$$

should be added to the two of Cayley.

We derive at once the absolute invariant

$$4(x_2\xi_1-\Omega_1)(x_1\xi_2-\mathrm{O}_1)+\{x_1\xi_1-x_2\xi_2-(n\mathrm{I}-2\mathrm{W})\}^2,$$

wherein the multiplications are algebraical and not at all operational.

The symbolic form of this invariant is

$$x_\xi^2+2na_\alpha^n x_\xi-n^2a_\alpha^n b_\beta^n-4na_\alpha^{n-1}a_x\alpha_\xi+2n^2a_\alpha^{n-1}b_\beta^{n-1}a_\beta b_\alpha;$$

where observe that the absence of determinant factors shows that it is absolute.

Observe also that we have already obtained the absolute invariant

$$4\Omega_1\mathrm{O}_1+(n\mathrm{I}-2\mathrm{W})^2,$$

whose symbolic form is $n^2a_\alpha^{n-1}b_\beta^{n-1}(2a_\beta b_\alpha-a_\alpha b_\beta)$.

This was obtained as the discriminant of the quadratic covariant

$$\begin{aligned}&x_1^2\Omega_1-x_1x_2(n\mathrm{I}-2\mathrm{W})-x_2^2\mathrm{O}_1\\ &\equiv a_x(x\alpha)a_\alpha^{n-1}.\end{aligned}$$

It might also have been obtained by taking the y-discriminant of the second polar of this covariant—

$$\begin{aligned}&y_1^2\Omega_1-y_1y_2(n\mathrm{I}-2\mathrm{W})-y_2^2\mathrm{O}_1\\ &\equiv a_y(y\alpha)a_\alpha^{n-1},\end{aligned}$$

and since the invariantive form

$$-(xy)y_\xi-a_y(y\alpha)a_\alpha^{n-1}$$

has the unsymbolic expression

$$y_1^2(x_2\xi_1-\Omega_1)-y_1y_2\{x_1\xi_1-x_2\xi_2-(n\mathrm{I}-2\mathrm{W})\}-y_2^2(x_1\xi_2-\mathrm{O}_1),$$

we find the covariant under examination by taking its discriminant.

The Operator $(a\xi)^\pi a_x^{n-\pi}$.

9. This is of degree one and weight zero in the coefficients and of order $n-2\pi$ in the variables; and p may be any of the integers 1, 2, 3, ... n.

Writing $a_x^n = f$, when $\pi = 1$,

$$n(a\xi)\,a_x^{n-1} = \frac{df}{dx_1}\xi_2 - \frac{df}{dx_2}\xi_1\,;$$

and, the operand being any covariant $\phi_x^m = \phi$, we obtain

$$\frac{df}{dx_1}\frac{d\phi}{dx_2} - \frac{df}{dx_2}\frac{d\phi}{dx_1},$$

which is the Jacobian of f and ϕ.

Similarly, if ϕ and ψ be any two covariants of orders p and q in the variables, the operation of

$$p(\phi\xi)\,\phi_x^{p-1} \text{ upon } \psi_x^q$$

produces the Jacobian of ϕ and ψ, viz.,

$$\frac{d\phi}{dx_1}\frac{d\psi}{dx_2} - \frac{d\phi}{dx_2}\frac{d\psi}{dx_1}\,;$$

and the operation of $q(\psi\xi)\,\psi_x^{q-1}$ upon ϕ_x^p

produces

$$\frac{d\psi}{dx_1}\frac{d\phi}{dx_2} - \frac{d\psi}{dx_2}\frac{d\phi}{dx_1}\,;$$

viz., the same Jacobian with negative sign.

In other words, the first transvectant (überschiebung) of ϕ over ψ is obtained by the operation of

$$p(\phi\xi)\,\phi_x^{p-1} \text{ upon } \psi_x^q,$$

and interchange of the forms ϕ, ψ merely changes the sign of the result.

We may write the theorem in the form

$$\begin{aligned} &p(\phi\xi)\,\phi_x^{p-1}.\,\psi_x^q \\ &= -\,q(\psi\xi)\,\psi_x^{q-1}.\,\phi_x^p \\ &= pq(\phi\psi)\,\phi_x^{p-1}.\,\psi_x^{q-1}. \end{aligned}$$

Passing to the case $\pi = 2$, we find

$$\begin{aligned} &n(n-1)(a\xi)^2 a_x^{n-2} \\ &= \frac{d^2f}{dx_1^2}\xi_2^2 - 2\frac{d^2f}{dx_1dx_2}\xi_1\xi_2 + \frac{d^2f}{dx_2^2}\xi^2\,; \end{aligned}$$

and thus, if ϕ_x^p, ψ_x^q be any two covariants,

$$\begin{aligned} &p(p-1)(\phi\xi)^2\phi_x^{p-2}.\,\psi_x^q \\ &= \frac{d^2\phi}{dx_1^2}\cdot\frac{d^2\psi}{dx_2^2} - 2\frac{d^2\phi}{dx_1dx_2}\cdot\frac{d^2\psi}{dx_1dx_2} + \frac{d^2\phi}{dx_2^2}\cdot\frac{d^2\psi}{dx_1^2}, \\ &= p(p-1)\,q(q-1)(\phi\psi)^2\phi_x^{p-2}\psi_x^{q-2}, \end{aligned}$$

the second transvectant of ϕ over ψ.

In general, it is clear that

$$\frac{p\,!}{\pi\,!}(\phi\xi)^\pi\phi_x^{p-\pi}\,.\,\psi_x^q$$

$$=(-)^\pi\frac{q\,!}{\pi\,!}(\psi\xi)^\pi\psi_x^{q-\pi}\,.\,\phi_x^p,$$

$$=\frac{p\,!}{\pi\,!}\;\frac{q\,!}{\pi\,!}(\phi\psi)^\pi\,\phi_x^{p-\pi}\psi_x^{q-\pi};$$

or, as it may be otherwise written,

$$\frac{\pi\,!}{q\,!}(\phi\xi)^\pi\,\phi_x^{p-\pi}\,.\,\psi_x^q,$$

$$=(-)^\pi\frac{\pi\,!}{p\,!}(\psi\xi)^\pi\,\psi_x^{q-\pi}\,.\,\phi_x^p,$$

$$=(\phi\psi)^\pi\,\phi_x^{p-\pi}\psi_x^{q-\pi};$$

showing that the πth transvectant of two forms is obtainable by a pure operation upon either of the forms.

So far as the writer is aware, such a transvection has not hitherto been exhibited as the result of a pure operation. The importance of the operational invariant

$$(a\xi)^\pi\,a_x^{n-\pi}$$

is thus evident.